Quasi-stationary distributions for a model of populations adapting to a changing environment

Aurelien Velleret, E. Pardoux and M. Kopp,
aurelien.velleret@nsup.org

Aix*Marseille université
Initiative d'excellence

Introduction

In this elementary model for the adapatation of a population to changing environmental conditions, the main focus is to gain more insight in the contribution of the various mutations.
A first issue that will be highlighted in this poster is the existence of a critical value of environmental change beyond which adaptation is not sufficient to prevent large extinction.

Ecological aspects

$(N_t)_{t\geq 0}$: size of the population
$(X_t)_{t\geq 0}$: gap relative to the moving optimum
we assume that the growth rate only depends on this adaptation coordinate X_t
Environmental change :
for simplicity, translation of the optimum at constant speed v
compensated by the fixation of mutations in the population

Population adapting

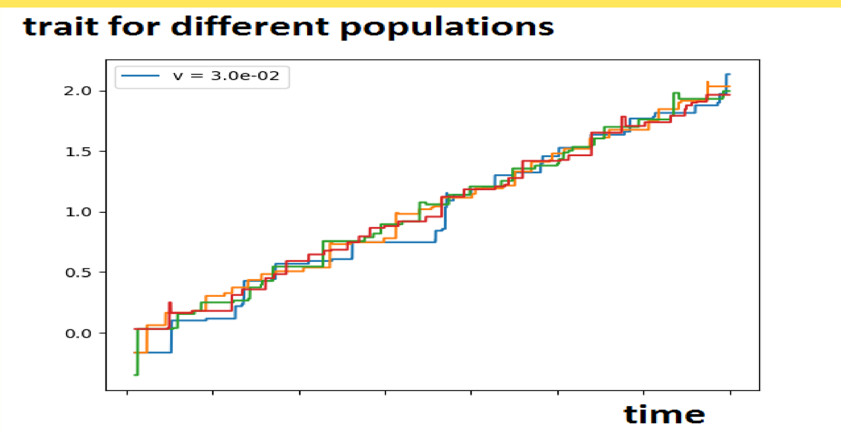

Figure 1: Following the moving optimum

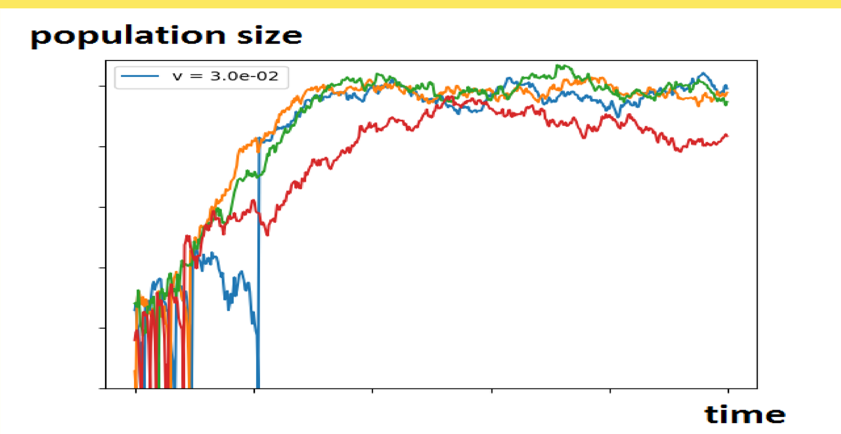

Figure 2: Stabilization of the population size

Conclusion and Perspectives

Intuitively, we expect to have some unique QSD when adaptation is quick to compensate the change in order to avoid extinction. Our proof relies however on the fact that even among surviving populations, profiles too largely unviable cannot be maintained for a long time. The critical speed is in a way the frontier between these two regimes of adaptation.
Another interesting issue is to understand how the "natural" profile of new mutations – in one individual – is biased when we look at the effects of mutations in the past of a typical surviving population.

References

[1] N. Champagnat and D. Villemonais. Exponential convergence to quasi-stationary distribution and q-process. *Probability Theory and Related Fields*, 164(1):243–283, Feb 2016.

[2] M. Kopp and J. Hermisson. The genetic basis of phenotypic adaptation ii: The distribution of adaptative substitutions of the moving optimum model. *Genetics*, 183:1453–1476, 2009.

[3] E. Nassar. *Modeles probabilistes de l'evolution d'une population dans un environnement variable.* PhD thesis, 2016. dirigee par Pardoux E. et Kopp M. AMU 2016.

Formalization of the stochastic process

Extinction happens at time τ_∂ as soon as N_t reaches 0 (no more individuals in the population),

$$\text{with} \qquad N_t = n + \int_0^t \left(r(X_s)\, N_s - c_p\, (N_s)^2\right) ds + \sigma \int_0^t \sqrt{N_s}\, dB_s,$$

where B is a Brownian Motion and $r(x) \underset{|x|\to\infty}{\longrightarrow} -\infty$ (extreme values of X_t are not viable)
General model for adaptation :

$$X_t = x - v\, t + \int_{[0,t]\times\mathbb{R}^d\times\mathbb{R}_+} w\, \mathbf{1}_{\{u \leq f(N_s)\, g(X_{s-},\, w)\}}\, M(ds, dw, du),$$

where M is a Poisson Point Process of intensity $ds\, \nu(dw)\, du$.
In the case f is bounded ($g \leq 1$), our model can also be described this way :
Let $M = \{T_i\}_{i\geq 1}$ be a Poisson Point Process with intensity $\|f\|_\infty$, ie inter-time are exponential r.v.
$(W_i)_{i\geq 1} \sim \nu(dw)$ be iid rv (effect of the mutations),
$(U_i)_{i\geq 1} \sim \mathcal{U}([0,1])$ be iid rv (filtering of the Point Process)

$$X_t = x - v\, t + \sum_{T_i \leq t} \mathbf{1}_{\left\{U_i \leq \frac{1}{\|f\|_\infty} f(N_{T_i})\, g(X_{T_i-}, W_i)\right\}} W_i$$

Existence and uniqueness of a quasi-stationary distribution

Convergence with exponential speed to a unique QSD α, uniformly over the initial condition :

$$\exists \zeta > 0,\ \forall \mu \in \mathcal{M}_1\left(\mathbb{R}\times\mathbb{R}_+^*\right),\ \exists C(\mu) > 0, \forall t > 0,$$

$$\left\|\mathbb{P}_\mu\left[(X,\, N)_t \in (dx,\, dn) \mid t < \tau_\partial\right] - \alpha(dx,\, dn)\right\|_{TV} \leq C(\mu)\, e^{-\zeta t}$$

Details of the proofs and the assumptions to appear shortly.
No assumption biologically unrelevant - except maybe $r(x) \underset{|x|\to\infty}{\longrightarrow} -\infty$ that cannot be observed !
This result holds for any value of v : no critical value can be obtained this way !

Simulation of the QSD

The random profile of a long-term surviving population :

$$\alpha(dx,\, dn) \sim \mathbb{P}_\mu\left[(X,\, N)_t \in (dx,\, dn) \mid t < \tau_\partial\right]$$

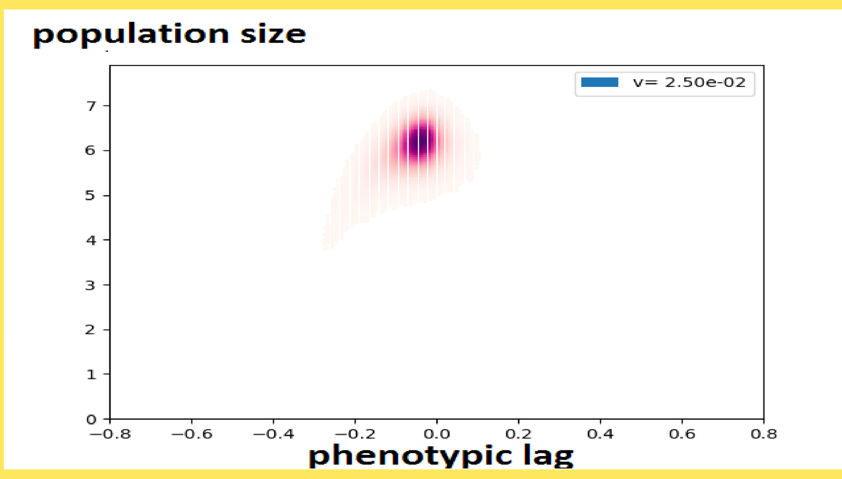

Figure 3: Well adapting population

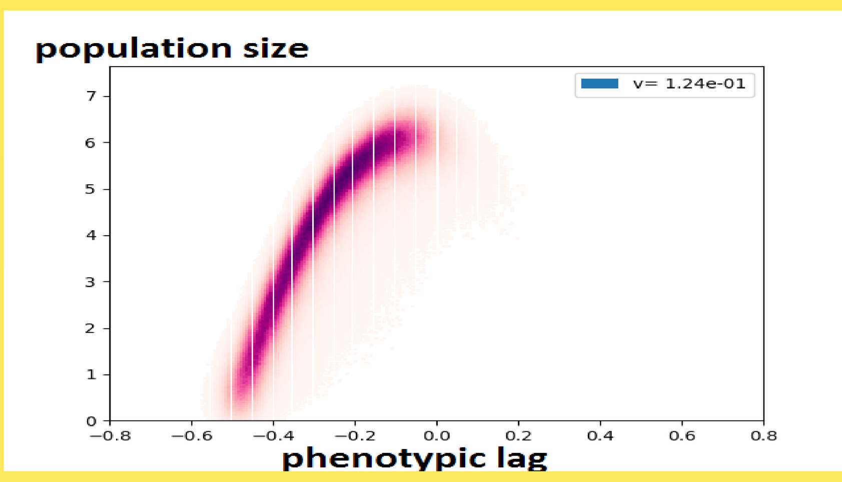

Figure 4: Limit of adaptation

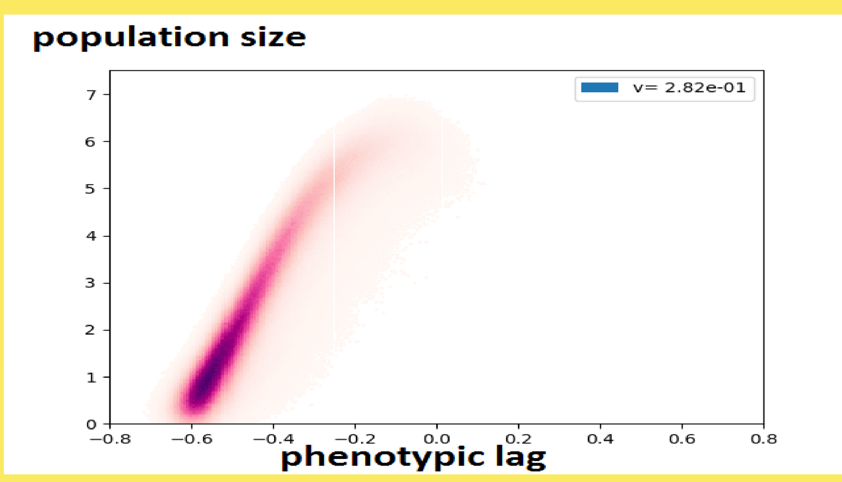

Figure 5: Adaptation compelled

Survival capacity

$$\eta(x, n) \sim \frac{\mathbb{P}_{(x,\,n)}(t < \tau_\partial)}{\mathbb{P}_\alpha(t < \tau_\partial)}$$

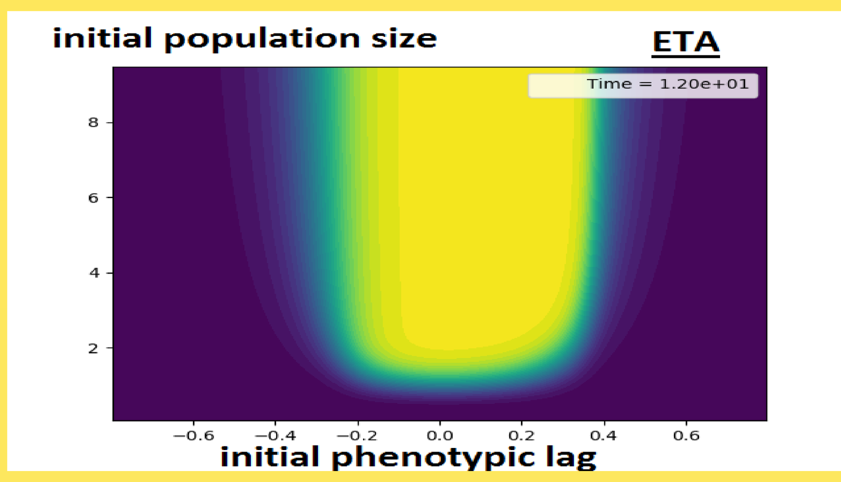

Figure 6: Well adapting population

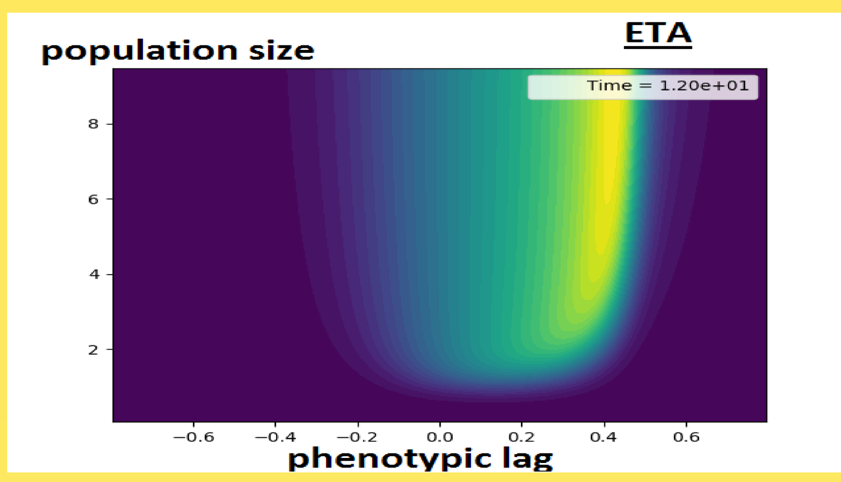

Figure 7: Limit of adaptation

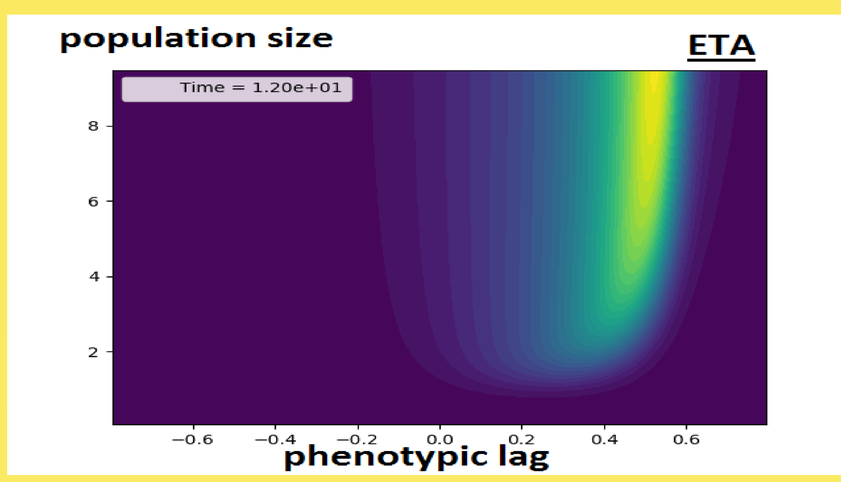

Figure 8: Adaptation compelled